Problem Set 3. Complex Networks

IMPORTANT: Follow the four steps in the solution of each problem, see the document 'Problem Set Rubrics'.

Problem 1. (Bounds on the maximal weight in GRG.) Assume that the weights $w_1, w_2, \ldots, w_n$ are deterministic. Show that Conditions 6.4 (a)-(b) imply that $\max_{1\le i\le n} w_i = o(n)$, and Conditions 6.4 (a)-(c) imply that $\max_{1\le i\le n} w_i = o(\sqrt{n})$. (When the weights w_i are random, these bounds hold in probability).

Hint: It is important to realize that the weights $w_1, w_2, \ldots, w_n$ may depend on n. In order to solve the problem, consider separately the weights that are smaller than x and bigger than x. Note that

$$\frac{1}{n}\max_{1\le i\le n} w_i \mathbf{1}\{w_i > x\} \le \frac{1}{n}\sum_{i=1}^{n} w_i \mathbf{1}\{w_i > x\}.$$

The result will follow from Conditions 6.4(a), 6.4(b), and the fact that $\lim_{x\to\infty} \mathbb{E}W\mathbf{1}\{W \le x\} = \mathbb{E}W$ by the definition of expectation (or by the Monotone Convergence Theorem).

Problem 2. (Dependent edges in GRG with i.i.d. weights.) Let the weights $\{w_i\}_{i=1}^n$ in the GRG_n graph be i.i.d. copies of a random variable W for which $\mathbb{E}W^2 < \infty$ (W does not depend on n). Assume further that $\mathbb{P}\{W \le \varepsilon\} = 0$ for some $\varepsilon > 0$. Prove that

$$n\mathbb{P}\{\text{edge (1,2) present}\} = n\mathbb{P}\{\text{edge (2,3) present}\} \to \mathbb{E}W \quad \text{as } n\to\infty,$$

and that

$$n^2\mathbb{P}\{\text{edges (1,2) and (2,3) present}\} \to \mathbb{E}W^2 \quad \text{as } n\to\infty.$$

Conclude that different edges that share a vertex are dependent whenever $\mathrm{Var}(W) > 0$.

Problem 3. Let X have a mixed Poisson distribution with mixing distribution F. Express the mean and variance of X through the moments of W, where W has distribution F.

Hint: For the mean, you can use the total expectation formula (or, the tower rule). Similarly, for the variance, you can use the law of total variance (please check the right formula, do not forget that it has two terms!). Direct calculations are possible as well. If you do direct calculations, you may assume that W has a continuous distribution with density f.

Problem 4. Let $n = 2$, $d_1 = 2$ and $d_2 = 4$. Use Proposition 7.7 to show that the probability that $\mathrm{CM}_n(\boldsymbol{d})$ consists of 3 self-loops equals 1/5.

Problem 5. Study the proof of Proposition 7.13. Reproduce the proof partially, to show that the number of self-loops S_n converges in distribution to a *Poisson* random variable with mean $\nu/2$. Explain each step and each formula that you use! Often explanations in the book are concise, e.g. (7.4.4), (7.4.9), (7.4.11), (7.4.13), (7.4.15), (7.4.16). You should apply these formulas to S_n only (everything related to M_n does not participate in your derivation, the formulas will become simpler), and explain why these formulas hold.

Problem 6. Fix a vertex $i \ge 1$ in the $\mathrm{PA}^{(1,\delta)}$ model. Show that $D_i(t) \to \infty$ almost surely, by using

$$\sum_{s=i}^{t} B_s \le_{\mathrm{st}} D_i(t),$$

where B_s are i.i.d. Bernoulli random variables with

$$\mathbb{P}\{B_s = 1\} = \frac{1+\delta}{s(2+\delta)+1+\delta}.$$

Hint: Apply the converse of the Borel-Cantelli lemma for independent events.

Problem 7. (Degree sequence in the uniform recursive tree.) Consider a uniform recursive tree: there are 2 vertices at time $t = 2$, and at time $t + 1$ a new vertex is added that forms exactly one edge with a uniformly picked old vertex. In this case, the theorem, which is identical to Theorem 8.3, still holds, with the proof along the same lines. However, due to the different attachment mechanism, the limiting distribution $(p_k)_{k\geq 1}$ is different. Compute $(p_k)_{k\geq 1}$ by writing down and solving the recursion similar to (8.6.11). Formulate a version of Theorem 8.3 for the uniform recursive tree.

Problem 8. Numerical assignment In this assignment we will numerically investigate the convergence of the degrees in GRG($\mathbf{w}$) with deterministic weights to a mixed Poisson distribution $Poisson(W)$, stated in Theorem 6.7 and Corollary 6.9(a). It is up to you how you approach this. If you want more specific instructions, you can follow the steps listed below.

1. Choose a probability distribution F of W. To make your experiment more realistic, choose a power law distribution, for example, a Pareto distribution. You may choose the parameter in such a way that the variance of W is infinite.

2. Choose the graph size n. For each vertex $1, 2, \ldots, n$, generate its weight. You can generate deterministic weights using the inverse distribution function. (Alternatively, you can generate the sequence of the weights $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ using independent realization of W, and then keep this sequence fixed for the given graph size n. Note that it follows from the Strong Law of Large Numbers that this sequence satisfies Condition 6.4(a,b) a.s. as $n \to \infty$).

3. Create edges as described in Section 6.2 using the weights $\mathbf{w}$. Repeat m times for the given graph size n. Choose a few (say, three) nodes i with different values of w_i. Compare the empirical distribution of the m realizations of D_i to the $Poisson(w_i)$ distribution.

4. In order to illustrate the general picture, make the following plot. Number the vertices in the increasing order of their weights: $w_1 \leq w_2 \leq \cdots \leq w_n$. On x-axis plot the number of a vertex: $1, 2, \ldots, n$. On y-axis show $m + 1$ plots: the value of w_i for each i (with one color), and the m realizations of D_i (with another color). Interpret this figure and compare to Corollary 6.9(a). (Note that if all weights are different, then $i = |\{k : w_k \leq w_i\}|$ so that in fact the first plot represents $F_n^{-1}(i/n)$, where F_n^{-1} is the inverse of the empirical distribution function F_n of $\mathbf{w}$).

5. Repeat the steps 2–4 for at least three different values of n.