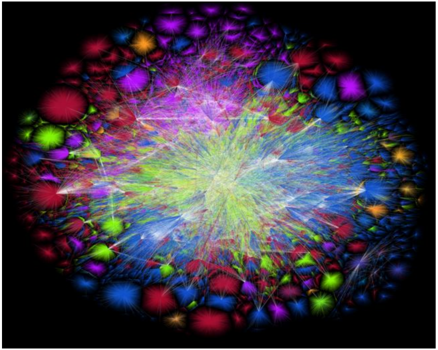
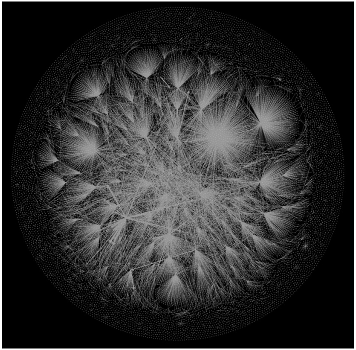
Complex Networks
Power laws

Nelly Litvak

Stochastic Operations Research & Data Science

Scale-free phenomenon





Left: Barrett Lyon / The Opte Project. Visualization of the routing paths of the Internet. 2015

Right: Tweets and retweets about Project X in Haren, 22-09-2012 5:00. Image by Marijn ten Thij

Scale-free graph sequences

- p_k is the fraction of nodes with degree k in a real-life network
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- Plots in the log-log scale: $p_k \approx const \cdot k^{-\tau}$

$$\log(p_k) \approx -\tau \log(k) + const$$

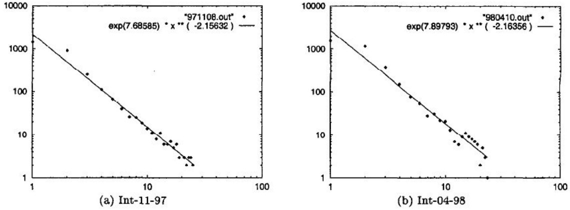
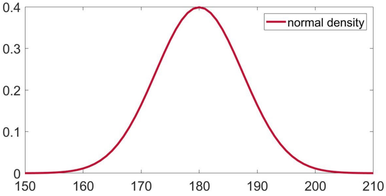


Figure 5: The outdegree plots: Log-log plot of frequency f_d versus the outdegree d.

Faloutsos, Faloutsos, Faloutsos (1999): degrees of Internet routers and autonomous systems.

‘Non-normal’


normal density
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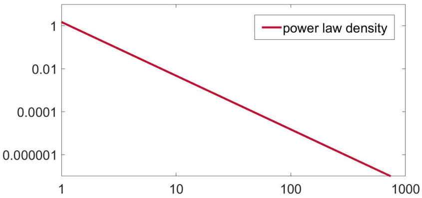
0.3
0.2
0.1
0
150
160
170
180
190
200
210


power law density
1
0.01
0.0001
0.000001
1
10
100
1000

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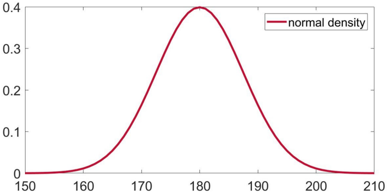

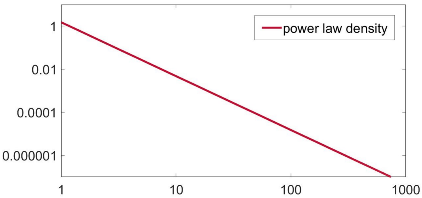

Length of a man	EU Web crawl 2015 1 070 557 254 webpages
Average Dutch man: 180cm Tallest man on Earth: 250cm	Average in-degree: 85.743 Maximal in-degree: 20 252 239

P. Boldi, A. Marino, M. Santini, and S. Vigna. BUbiNG: Massive crawling for the masses. WWW 2014

Complementary cumulative distribution function

- X is a discrete random variable, $p_k = P(X = k)$, $k = 0, 1, \dots$
- $\bar{F}(k) = P(X \geq k) = \sum_{s=k}^{\infty} p_s$
- Power law (assume discrete version of Pareto distribution): $p_k = ck^{-\tau}$, $k \geq k_0$ (**)
- For any $x \geq k_0$

$$\int_k^{\infty} cx^{-\tau}\,dx \leq \sum_{s=k}^{\infty} cs^{-\tau} \leq \int_{k-1}^{\infty} cx^{-\tau}\,dx$$

$$\int_k^{\infty} cx^{-\tau}\,dx = -\frac{c}{\tau-1}x^{-\tau+1}\Big|_k^{\infty} = \frac{c}{\tau-1}k^{-\tau+1}$$

- $\bar{F}(k) \approx Ck^{-\tau+1}$ is less restrictive than (**). $\bar{F}(k)$ is used to represent the distribution of the data

Regular variation

$X \geq 0$ -regularly varying with exponent $\tau - 1$

$$P(X > x) = L(x)x^{-(\tau-1)}, \quad x \geq 0$$

$L(x)$ -Slowly varying function

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1, \quad \text{for all } t > 0$$

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Lemma. Let L be slowly varying function. Then for all $\epsilon > 0$ there exists $T > 0$ such that, for $x > T$

$$x^{-\epsilon} \leq L(x) \leq x^{\epsilon}.$$

Moments of scale-free distribution

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$$Var(X) = E(X^2) - (E(X))^2 < \infty \quad \text{only if } \tau > 3$$

- $\tau > 4$ is also an important 'nice' special case
- Many real networks have $\tau \in (2, 4)$

Maximal degree

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- Compare to the maximum degree in the web crawl on slide 4

Extreme value theory

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Theorem (alternative version of Theorem 2.33)

Let $(X_n))n \geq 0$ be a sequence of i.i.d. regularly varying random variables for some $\tau > 1$. Then for any fixed k

$$\frac{1}{u_n}(X_{(n+1-i)})_{i=1}^{k} \xrightarrow{d} (\xi_i)_{i=1}^{k},$$

where $\xi_i = \Gamma_i^{-1/(\tau-1)}, \quad \Gamma_i = E_1 + \cdots + E_i$, and $E_1, E_2, \ldots$ are independent exponential random variables with mean 1. In particular (Theorem 2.30), ξ_1 has a Fréchet distribution with parameter $\alpha = \tau - 1$

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- $k/n \to 0, \; k \to \infty$
- How to choose k?

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- Social networks, collaboration graphs
- Power law with exponential cut-off

$$p_k = ck^{-\tau} e^{-k/A}, \quad k \geq 1.$$

When A is very large then the exponent influences only the probabiltiy of the high values of k.